On Ishikawa Iteration with Different Control Conditions for Asymptotically Non-expansive Non-self Mappings

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Introduction

Definition 1.1

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A map $T : C \to C$ is said to be asymptotically nonexpansive ([2]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

(1.1) $\|T^n x - T^n y\| \le k_n \|x-y\|$

for all x, y $\in$ C and n $\ge$ 1.

T is said to be uniformly L-Lipschitzian ([2]) if

$$\|T^n x - T^n y\| \le L\|x-y\|$$

for all x, y $\in$ C and n $\ge$ 1, where L is a positive constant.

For a map of T of C into itself, the Ishikawa iteration scheme is studied ([1]): $x_1 \in C$, and

(1.2) $$\begin{cases} x_{n+1} = \alpha_n T^n(y_n) + (1-\alpha_n)x_n \\ y_n = \beta_n T^n(x_n) + (1-\beta_n)x_n \end{cases}$$

Definition 1.2

Let C be a nonempty closed convex subset of a Banach space X and let D be a nonempty subset of C. A retraction from C to D is a mapping P : C $\to$ D such that Px = x for x $\in$ D. A retraction P from C to D is nonexpansive if P is nonexpansive (i.e., $\|Px-Py\| \le \|x-y\|$ for x, y $\in$ C).

Let E be a real normed linear space, K a nonempty subset of E. Let P : E $\to$ K be the nonexpansive retraction of E onto K. A map T : K $\to$ E is said to be asymptotically nonexpansive([2]) if there exists a sequence $(k_n) \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that the following inequality holds.

(1.3) $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n\|x-y\|, \forall x, y \in K, n \ge 1.$

T is called uniformly L-Lipschitzian if there exists L > 0 such that

(1.4) $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le L\|x-y\|, \forall x, y \in K, n \ge 1.$

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. The following iteration scheme is studied:

(1.5) $$x_1 \in K, \begin{cases} x_{n+1} = P(\alpha_n T(PT)^{n-1}(y_n) + (1-\alpha_n)x_n) \\ y_n = P(\beta_n T(PT)^{n-1}(x_n) + (1-\beta_n)x_n) \end{cases}$$

Lemma1.1 ([6]) Let r > 0 be a fixed real number then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex map $g : [0,\infty) \to [0,\infty)$ with

$g(0) = 0$ such that for all $x, y \in Br[0] = \{x \in E : \|x\| \le r\}$,

$\|\lambda x + (1-\lambda)y\|2 \le \lambda \|x\|2 + (1-\lambda) \|y\|2-\lambda(1-\lambda)g (\|x-y\|)$ for all $\lambda \in [0,1]$

Lemma 1.2 ([7]) Let $g:[0,\infty) \to [0,\infty)$ with $g(0) = 0$ be a strictly increasing map. If a sequence $\{x_n\}$ in $[0, \infty)$ satisfies $\lim_{n\to\infty} g(x_n) = 0$, then $\lim_{n\to\infty} (x_n) = 0$.

2. Main Results

Lemma 2.1

Let E be a real uniformly convex Banach space, K closed convex nonempty subset of E. Let $T : K \to E$ be asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ such that

$\sum_{n\ge1} k_n - 1 < \infty$ and $F(T) \ne \phi$. Let $\{\alpha_n\} \subset (0,1)$ be such that $\varepsilon \le 1 - \alpha_n \le 1- \varepsilon \ \forall n \ge 1$ and some $\varepsilon > 0$. From arbitrary $x_1 \in k$ define a sequence $\{x_n\}$ by equation (1.5). Then $\lim_{n\to\infty}\|x_n - x^*\|$ exists for each $x^* \in F(T)$.

Proof

For any $x^* \in F(T)$, utilizing (1.5), we have

$\|x_{n+1}-x^*\| = \|P(\alpha_n T (PT)^{n-1} (y_n) + (1-\alpha_n) x_n)-Px^*)\|$

$= \|\alpha_n(T(PT)^{n-1} (y_n)-T(PT)^{n-1} x^*) + (1-\alpha_n) (x_n-x^*)\|$

$\le \alpha_n k_n \|y_n-x^*\| + (1-\alpha_n) \|x_n-x^*\|$

$\le \alpha_n k_n\|\beta_n (T(PT)^{n-1} (x_n)-x^*) + (1-\beta_n) (x_n-x^*)\| + (1-\alpha_n) \|x_n-x^*\|$

$\le \alpha_n\beta_n k_n\| T(PT)^{n-1} (x_n)-x^*\| + \alpha_n k_n (1-\beta_n) \| x_n-x^*\| + (1-\alpha_n) \| x_n-x^* \|$

$\le \alpha_n\beta_n k_n^2 \| x_n-x^*\| + \alpha_n k_n (1-\beta_n) \| x_n-x^*\| + (1-\alpha_n) \| x_n-x^* \|$

$\le \| x_n-x^* \| [\alpha_n\beta_n k_n^2 + \alpha_n k_n-\alpha_n\beta_n k_n + 1-\alpha_n]$

$\leq \| x_n - x^* \| \, [\alpha_n \beta_n k_n (x_n - 1) + \alpha_n (k_n - 1) + 1]$

$\leq \| x_n - x^* \| \, [1 + \mu_n]$, where $\mu_n = \alpha_n \beta_n k_n (k_{n-}1) + \alpha_n (k_n - 1)$

$\|x_{n+1} - x^*\| \leq \|x_{n-}x^*\| \, (1 + \mu_1)\, (1 + \mu_2) \ldots (1 + \mu_n)$

$\leq \| x_n - x^* \| \, e^{\sum_{n=1}^{\infty} \mu_i}$

since, $e^{\sum_{n=1}^{\infty} \mu_i}$, $\|x_n - x^*\|$ is bounded.

This proves that $\|x_n - x^*\|$ is a bounded sequence and hence $\lim_{n \to \infty} \|x_n - x^*\|$ exists.

Theorem 2.2

Let E be a real uniformly convex Banach space, K closed convex nonempty subset of E. Let T : K $\to$E be asymptotically nonexpansive with sequence

$\{k_n\} \subset [1, \infty)$ such that $\sum_{n \geq 1} k_n - 1 < \infty$ and $F(T) \neq \phi$. Let $\{\alpha_n\} \subset (0,1)$

be such that

$\varepsilon \leq 1 - \alpha_n \leq 1 - \varepsilon \;\forall\; n \geq 1$ and some $\varepsilon > 0$. From arbitrary $x_1 \in k$ define a sequence $\{x_n\}$ by equation(1.5). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,1] and satisfy the following condition :

$$\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty, \limsup_{n \to \infty} \beta_n < 1 \qquad \textbf{(2.1)}$$

Then $\liminf_{n \to \infty} \| x_n - T(PT)^{n-1} x_n \| = 0$.

Proof

For any $x^* \in F(T)$, utilizing (1.5), we have

$\|x_{n+1} - x^*\|^2 = \|\alpha_n (T(PT)^{n-1}(y_n) - x^*) + (1-\alpha_n)(x_n - x^*) \|^2$

$\leq \alpha_n \| T(PT)^{n-1} (y_n) - x^*\|^2 + (1-\alpha_n) \|x_n - x^*\|^2 - \alpha_n (1-\alpha_n) \, g (\|x_n - (T(PT)^{n-1} (y_n) \|)$

$\leq \alpha_n k_n \| y_n - x^*\| + (1-\alpha_n) \| x_n - x^*\|^2 - \alpha_n (1-\alpha_n) g (\|x_n - (T(PT)^{n-1} (y_n) \|)$

$\leq \alpha_n k_n \| \beta_n (T(PT)^{n-1} (x_n) - x^*) + (1 - \beta_n) (x_n - x^*)\|^2 + (1 - \alpha_n) \|(x_n - x^*)\|^2$

$-\alpha_n (1 - \alpha_n) \, g (\|x_n - T(PT)^{n-1} (y_n) \|)$

$\leq \alpha_n k_n^2 \beta_n \|x_n - x^*\|^2 + \alpha_n k_n (1 - \beta_n) \|x_n - x^*\|^{2-} \alpha_n k_n \beta_n (1-\beta_n) \, g (\|x_n - T(PT)^{n-1}(x_n)\|$

$+ (1-\alpha_n) \|x_n - x^*\|^2 - \alpha_n (1-\alpha_n) \, g (\|x_n - T(PT)^{n-1} (y_n)\|)$

$\|x_{n+1}-x^*\|^2 \leq \|x_n-x^*\|^2 + \{\alpha_n\beta_n k_n^2 + \alpha_n k_n - \alpha_n\beta_n k_n - \alpha_n\} \|x_n-x^*\|^2$

$- \alpha_n\beta_n k_n (1-\beta_n)$ g $(\|x_n-T(PT)^{n-1}(x_n)\| - \alpha_n (1-\alpha_n)$g $(\|x_n-T^n(PT)^{n-1}(y_n)\|$

$$\alpha_n(1-\alpha_n) \text{ g } (\|x_n-T(PT)^{n-1}(y_n)\| \leq \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \{\alpha_n\beta_n k_n (k_n-1) + \alpha_n(k_n-1)\} \|x_n-x^*\|^2 \tag{2.2}$$

$$\alpha_n\beta_n k_n (1-\beta_n) \text{ g } \|x_n-T(PT)^{n-1}(x_n)\|) \leq \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \{\alpha_n\beta_n k_n (k_n-1) + \alpha_n(k_n-1)\} \|x_n-x^*\|^2 \tag{2.3}$$

$\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (2.1)

Let m $\geq$ 1. Then from the inequality (2.2), we have

$$\sum_{n=1}^{m} \alpha_n (1-\alpha_n) \text{ g } (\|x_n-T(PT)^{n-1}(y_n)\| \leq \|x_1-x^*\|^2 - \|x_{m+1}-x^*\|^2$$

$$+ \sum_{n=1}^{m} \{\alpha_n\beta_n k_n (k_n-1) + \alpha_n(k_n-1)\}\|x_n-x^*\|^2$$

when m $\rightarrow \infty$, we have

$$\sum_{n=1}^{m} \alpha_n (1-\alpha_n) \text{ g } (\|x_n-T(PT)^{n-1}(y_n)\| < \infty$$

since $\sum_{n=1}^{\infty} (k_n-1) < \infty$, $\|x_n - x^*\|$ is bounded and $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$

$$\therefore \liminf_{n\rightarrow\infty} \text{ g } \|x_n-T(PT)^{n-1}(y_n)\| = 0$$

From lemma 1.3 we get

$$\liminf_{n\rightarrow\infty} \|x_n-T(PT)^{n-1}(y_n)\| = 0$$

since $\|x_n-T(PT)^{n-1}(x_n)\| = \|x_n-T(PT)^{n-1}(y_n) + T(PT)^{n-1}(y_n)-T(PT)^{n-1}(x_n)\|$

$\leq \|x_n-T(PT)^{n-1}(y_n)\| + \| T(PT)^{n-1}(x_n)-T(PT)^{n-1}(y_n)\|$

$\leq \|x_n-T(PT)^{n-1}(y_n)\| + k_n \|x_n-y_n\|$

$\leq \|x_n-T(PT)^{n-1}(y_n)\| + k_n \|x_n-\beta_n T(PT)^{n-1} x_n x_n + \beta_n x_n \|$

$\leq \|x_n - T(\mathrm{PT})^{\mathrm{n-1}} (y_n)\| + k_n\beta_n \|x_n - T(\mathrm{PT})^{\mathrm{n-1}} (x_n)\|$

$(1-k_n\beta_n) \|x_n - T(\mathrm{PT})^{\mathrm{n-1}}(x_n)\| \leq \|x_n - T(\mathrm{PT})^{\mathrm{n-1}} (y_n)\|$

$\liminf (1-k_n\beta_n) > 0$ & $\liminf_{n\to\infty} \|x_n - T(\mathrm{PT})^{\mathrm{n-1}n} (y_n)\| = 0$

$\liminf_{n\to\infty} \|x_n - T(\mathrm{PT})^{\mathrm{n-1}}(x_n)\| = 0$

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